

Convergence Rate of Zero Viscosity Limit on Large Amplitude Solution to a Conservation Laws Arising in Chemotaxis

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Abstract

In this paper, we investigate large amplitude solutions to a system of conservation laws which is transformed, by a change of variable, from the well-known Keller-Segel model describing cell (bacteria) movement toward the concentration gradient of the chemical that is consumed by the cells. For the Cauchy problem and initial-boundary value problem, the global unique solvability is proved based on the energy method. In particular, our main purpose is to investigate the convergence rates as the diffusion parameter ε goes to zero. It is shown that the convergence rates in L^∞ -norm are of the order $O(\varepsilon)$ and $O(\varepsilon^{3/4})$ corresponding to the Cauchy problem and the initial-boundary value problem respectively.

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1 Introduction

In this paper, we investigate large amplitude solutions to a system of conservation laws

$$\begin{cases} u_t^\varepsilon + \left(\varepsilon (u^\varepsilon)^2 - v^\varepsilon \right)_x = \varepsilon u_{xx}^\varepsilon, \\ v_t^\varepsilon - (u^\varepsilon v^\varepsilon)_x = v_{xx}^\varepsilon, \end{cases} \quad (1.1)$$

which is transformed, by a change of variable, from the well-known Keller-Segel model describing cell (bacteria) movement toward the concentration gradient of the chemical that is consumed by the cells. Here $\varepsilon > 0$ is a positive constant.

The aim here is to study the global unique solvability on the Cauchy problem and initial-boundary value problem in the framework of large-amplitude H^2 solutions. In particular, our main purpose is to investigate the convergence rates as the diffusion parameter ε goes to zero. It is shown that the convergence rates in L^∞ -norm are of the order $O(\varepsilon)$ and $O(\varepsilon^{3/4})$ corresponding to the Cauchy problem and the initial-boundary value problem respectively.

Firstly we are concerned with the Cauchy problem of (1.1) with initial data

$$(u^\varepsilon, v^\varepsilon)(x, 0) = (u_0^\varepsilon(x), v_0^\varepsilon(x)) \rightarrow (0, v_\infty) \text{ as } x \rightarrow \pm\infty. \quad (1.2)$$

Hereafter, v_∞ is a given positive constant. On one hand, the global existence of solutions to the Cauchy problem (1.1), (1.2) for any fixed $\varepsilon > 0$ is shown. On the other hand, we prove that solutions to the Cauchy problem (1.1), (1.2) convergence to solutions to the problem of zero viscosity limit when $\varepsilon \rightarrow 0^+$, which is formally formulated as follows:

$$\begin{cases} u_t^0 - v_x^0 = 0, \\ v_t^0 - (u^0 v^0)_x = v_{xx}^0, \end{cases} \quad (1.3)$$

with initial data

$$(u^0, v^0)(x, 0) = (u_0^0(x), v_0^0(x)) \rightarrow (0, v_\infty) \text{ as } x \rightarrow \pm\infty. \quad (1.4)$$

In addition, we consider the initial-boundary value problem of (1.1) with initial data

$$(u^\varepsilon, v^\varepsilon)(x, 0) = (u_0^\varepsilon(x), v_0^\varepsilon(x)), \quad 0 \leq x \leq 1, \quad (1.5)$$

and the boundary conditions

$$(u^\varepsilon, v_x^\varepsilon)(0, t) = (u^\varepsilon, v_x^\varepsilon)(1, t) = (0, 0), \quad t \geq 0, \quad (1.6)$$

which implies

$$u_{xx}^\varepsilon(0, t) = u_{xx}^\varepsilon(1, t) = 0, \quad t \geq 0, \quad (1.7)$$

where the compatibility conditions $u_0^\varepsilon(0) = u_0^\varepsilon(1) = 0$ and $v_{0x}^\varepsilon(0) = v_{0x}^\varepsilon(1) = 0$ are satisfied.

For one thing, the global existence of solutions to the initial-boundary value problem of (1.1), (1.5), (1.6) for any fixed $\varepsilon > 0$ is shown. For another thing, we prove that solutions to the

initial-boundary value problem of (1.1), (1.5), (1.6) convergence to solutions to the problem of zero viscosity limit when $\varepsilon \rightarrow 0^+$, which is formally formulated as follows:

$$\begin{cases} u_t^0 - v_x^0 = 0, \\ v_t^0 - (u^0 v^0)_x = v_{xx}^0, \end{cases} \quad (1.8)$$

with initial data

$$(u^0, v^0)(x, 0) = (u_0^0(x), v_0^0(x)), \quad 0 \leq x \leq 1, \quad (1.9)$$

and the boundary conditions

$$u^0(0, t) = u^0(1, t) = 0, \quad t \geq 0, \quad (1.10)$$

which implies

$$v_x^0(0, t) = v_x^0(1, t) = 0, \quad t \geq 0, \quad (1.11)$$

where the compatibility conditions $u_0^0(0) = u_0^0(1) = 0$ and $v_{0x}^0(0) = v_{0x}^0(1) = 0$ are satisfied.

Precisely speaking, these results are given as follows. We have the first main theorem, which is concerned with the global existence of the Cauchy problem (1.1)-(1.2).

Theorem 1.1 (Global existence) *Let $(u_0^\varepsilon, v_0^\varepsilon - v_\infty) \in H^2(\mathbf{R})$ and assume that there exists a positive constant $\alpha > 0$ such that $\inf_{x \in \mathbf{R}} v_0^\varepsilon(x) \geq \alpha$. Then there exists a unique global solution $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ of the Cauchy problem (1.1)-(1.2) which satisfies*

$$\begin{cases} (u^\varepsilon, v^\varepsilon - v_\infty) \in L^\infty([0, \infty), H^2(\mathbf{R})), \\ \frac{v_x^\varepsilon}{\sqrt{v^\varepsilon}} \in L^2([0, \infty), L^2(\mathbf{R})), \quad v_{xx}^\varepsilon \in L^2([0, \infty), H^1(\mathbf{R})) \end{cases} \quad (1.12)$$

and

$$\begin{aligned} & \|u^\varepsilon(t)\|_{H^2}^2 + \|v^\varepsilon(t) - v_\infty\|_{H^2}^2 + \varepsilon \int_0^t \|u_x^\varepsilon(s)\|_{H^2}^2 ds \\ & + \int_0^t \left\{ \left\| \frac{v_x^\varepsilon(s)}{\sqrt{v^\varepsilon}} \right\|_{L^2}^2 + \left\| \sqrt{v^\varepsilon} u_x^\varepsilon(s) \right\|_{L^2}^2 + \|v_{xx}^\varepsilon(s)\|_{H^1}^2 \right\} ds \\ & \leq C \left(\|u_0^\varepsilon\|_{H^2}^2 + \|v_0^\varepsilon - v_\infty\|_{H^2}^2 \right). \end{aligned} \quad (1.13)$$

Moreover, for any fixed $T > 0$, there exists a positive constant $C(T)$ which depends only on T such that

$$v^\varepsilon(x, t) \geq C(T). \quad (1.14)$$

Here C and $C(T)$ are all positive constants independent of ε .

Remark 1.1. In the case of $\varepsilon = 0$, the global unique solvability on the Cauchy problem (1.3)-(1.4) was included in [5], which will be summarized in Lemma 2.4 later. As is well-known that it is more difficult when the nonlinearity is higher in the setting of large amplitude. It is nontrivial to obtain the estimate on (1.13) in the setting of large amplitude for the case of $\varepsilon > 0$ since

the nonlinear term $\{(\varepsilon u^\varepsilon)^2\}_x$ appears. We overcome these difficulties by applying some suitable interpolation inequalities.

Next, the following main result refers to convergence rate under the additional assumptions on initial data.

Theorem 1.2 (L^∞ -convergence rate) *Under the same conditions of Theorem 1.1, we assume the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ and (u_0^0, v_0^0) satisfy $\|u_0^\varepsilon - u_0^0\|_{H^2(\mathbf{R})} + \|v_0^\varepsilon - v_0^0\|_{H^2(\mathbf{R})} = O(\varepsilon)$, and the further regularity $(u_0^0, v_0^0) \in H^3(\mathbf{R})$. Then we have*

$$\|(u^\varepsilon - u^0)(t)\|_{L^\infty(\mathbf{R})} + \|(v^\varepsilon - v^0)(t)\|_{L^\infty(\mathbf{R})} \leq C\varepsilon. \quad (1.15)$$

Here $C > 0$ is a positive constant independent of ε .

Remark 1.2. In the case of $\varepsilon > 0$, for the initial-boundary value problem of (1.1) with Dirichlet boundary conditions $u^\varepsilon(0, t) = u^\varepsilon(1, t) = 0$, $v^\varepsilon(0, t) = v^\varepsilon(1, t) = 0$, with the help of the methods in [25], we can prove the global unique solvability in the setting of small amplitude since the difficulties derived from the boundary effect on the viscosity $\varepsilon u_{xx}^\varepsilon$. What's more, the boundary layer will occur and the boundary layer thickness will be shown as in [4, 7, 18]. These are left to our future study since these are not in accordance with our main aim in the paper, which mainly refers to large amplitude solution.

Our final result focusing on the initial-boundary value problem is stated as follows.

Theorem 1.3 (Initial-boundary value problem) *Let $(u_0^\varepsilon, v_0^\varepsilon) \in H^2([0, 1])$ and assume that there exists a positive constant $\alpha > 0$ such that $\inf_{x \in [0, 1]} v_0^\varepsilon(x) \geq \alpha$.*

(i) Then there exists a unique global solution $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ of the initial-boundary value problem (1.1), (1.5) and (1.6) which satisfies

$$\begin{cases} (u^\varepsilon, v^\varepsilon) \in L^\infty([0, \infty), H^2([0, 1])), \\ \frac{v_x^\varepsilon}{\sqrt{v^\varepsilon}} \in L^2([0, \infty), L^2([0, 1])), \quad v_{xx}^\varepsilon \in L^2([0, \infty), H^1([0, 1])) \end{cases} \quad (1.16)$$

and

$$\begin{aligned} & \|u^\varepsilon(t)\|_{H^2}^2 + \|v^\varepsilon(t)\|_{H^2}^2 + \varepsilon \int_0^t \|u_x^\varepsilon(s)\|_{H^2}^2 ds \\ & + \int_0^t \left\{ \left\| \frac{v_x^\varepsilon(s)}{\sqrt{v^\varepsilon}} \right\|_{L^2}^2 + \left\| \sqrt{v^\varepsilon} u_x^\varepsilon(s) \right\|_{L^2}^2 + \|v_{xx}^\varepsilon(s)\|_{H^1}^2 \right\} ds \\ & \leq C \left(\|u_0^\varepsilon\|_{H^2}^2 + \|v_0^\varepsilon\|_{H^2}^2 \right). \end{aligned} \quad (1.17)$$

Moreover, for any fixed $T > 0$, there exists a positive constant $C(T)$ which depends only on T such that

$$v^\varepsilon(x, t) \geq C(T). \quad (1.18)$$

(ii) We assume the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ and (u_0^0, v_0^0) satisfy $\|u_0^\varepsilon - u_0^0\|_{H^2([0, 1])} + \|v_0^\varepsilon - v_0^0\|_{H^2([0, 1])} = O(\varepsilon)$. Then we have

$$\|(u^\varepsilon - u^0)(t)\|_{L^\infty([0, 1])} + \|(v^\varepsilon - v^0)(t)\|_{L^\infty([0, 1])} \leq C\varepsilon^{\frac{3}{4}}. \quad (1.19)$$

Here C and $C(T)$ are all positive constants independent of ε .

Remark 1.3. Convergence rates (1.19) on the initial-boundary value problem is slower than one in (1.15) on the Cauchy problem, which is caused by the boundary effect. Essentially it is different between initial-boundary value problem and Cauchy problem that the regularity of solutions to the Cauchy problem can be improved when the regularity of initial data is imposed further regularity whereas it fails for the initial-boundary value problem.

The problem of the zero viscosity limit is one of the important topics. In particular, when parabolic equations with small viscosity are applied as perturbations, convergence rate of the Cauchy problem or the boundary layer question of the initial-boundary value problem for many other equations also arises in the theory of hyperbolic systems in the case of one-dimension or multi-dimension, cf. [3, 4, 7, 18, 22, 23]. To our knowledge, fewer results on the equations (1.1) have been obtained in this direction.

Now let us review some known results related to the system (1.1) and (1.3), which has been extensively studied by several authors in different contexts, cf. [5, 11, 14, 25]. The conservation laws (1.1) are derived from the original well-known Keller-Segel model

$$\begin{cases} c_t = \varepsilon c_{xx} - uf(c), \\ u_t = (Du_x - \chi uc^{-1}c_x)_x, \end{cases} \quad (1.20)$$

which was proposed by Keller and Segel in [8] to describe the traveling band behavior of bacteria due to the chemotactic response (i.e., the oriented movement of cells to the chemical concentration gradient) observed in experiments [1, 2]. In model (1.20), $u(x, t)$ and $c(x, t)$ denote the cell density and the chemical concentration, respectively. $D > 0$ is the diffusion rate of cells (bacteria) and $\varepsilon > 0$ is the diffusion rate of chemical substance. χ is a positive constant often referred to as chemosensitivity. $f(c)$ is a kinetic function describing the chemical reaction between cells and the chemical.

When $f(c)$ is a positive constant, namely, $f(c) = \alpha > 0$, the existence of traveling wave solutions of (1.20) with $\varepsilon = 0$ was established by Keller and Segel themselves in [8]. When $\varepsilon \neq 0$, the existence and linear instability of traveling wave solutions of (1.20) were shown by Nagai and Ikeda in [15] where the authors also obtained the diffusion limits of traveling wave solutions of (1.20) as ε approaches zero. Precisely they proved that the traveling wave solution in the form $(B^\varepsilon(x, t), S^\varepsilon(x, t)) = (B^\varepsilon(z), S^\varepsilon(z))$ ($z = x - ct$) approximates to the corresponding traveling wave solution $(B^0(x, t), S^0(x, t)) = (B^0(z), S^0(z))$ ($z = x - ct$) when ε goes to zero. For the reduction of Keller-Segel system (1.20) to system (1.1), we refer to Refs. [14], see also Appendix in this paper.

In [14], Li and Wang established the existence and the nonlinear stability of traveling wave solutions to a system of conservation laws (1.1). They prove the existence of traveling fronts by the phase plane analysis and show the asymptotic nonlinear stability of traveling wave solutions without the smallness assumption on the wave strengths by the method of energy estimates.

There is limiting case of the Keller-Segel model (1.20). That is when the diffusion of chemical substance is so small that it is negligible, i.e., $\varepsilon \rightarrow 0^+$, then the model (1.20) becomes

$$\begin{cases} c_t = -uf(c), \\ u_t = (Du_x - \chi uc^{-1}c_x)_x. \end{cases} \quad (1.21)$$

A version of system (1.21) was proposed by Othmer and Stevens in [16] to describe the chemotactic movement of particles where the chemicals are non-diffusible. Othmer and Stevens in [16] have developed a number of mathematical models of chemotaxis to illustrate aggregation leading (numerically) to nonconstant steady-states, blow-up resulting in the formation of singularities and collapse or the formation of a spatially uniform steady state. The models developed in [16] have been studied in depth by Levine and Sleeman in [9]. They gave some heuristic understanding of some of these phenomena and investigated the properties of solutions of a system of chemotaxis equation arising in the theory of reinforced random walks. Y. Yang, H. Chen and W.A. Liu in [24] studied the global existence and blow-up in a finite-time of solutions for the case considered in [9], respectively. They found that even at the same growth rate the behavior of the biological systems can be very different just because they started their action in different conditions. For the other results on the initial-boundary value problem of (1.21) refer to [6, 10].

Recently, the modified model related closely to chemotaxis was also investigated by Painter and Hillen in [17]. They explored the dynamics of a one-dimensional Keller-Segel type model for chemotaxis incorporating a logistic cell growth term.

Finally, we have to mention the work in [25, 5], which investigated the initial-boundary value problem and Cauchy problem on limit equation (1.3) and motivate our investigation in this paper. Zhang and Zhu in [25] studied the initial-boundary value problem of (1.3) with initial data

$$(u^0, v^0)(x, 0) = (u_0^0(x), v_0^0(x)) \quad (1.22)$$

and boundary conditions

$$u^0(0, t) = u^0(1, t) = 0. \quad (1.23)$$

Let $(u_0^0, v_0^0 - 1) \in H^2([0, 1])$ and assume that there exists a positive constant $\epsilon > 0$ sufficiently small such that $\|u_0^0\|_{H^2}^2 + \|v_0^0 - 1\|_{H^2}^2 \leq \epsilon$. Then there exists a unique global solution $(u^0(x, t), v^0(x, t))$ of the initial-boundary value problem (1.3), (1.22) and (1.23) which satisfies

$$(u^0, v^0) \in L^\infty([0, \infty), H^2([0, 1])), \quad v_x^0 \in L^2([0, \infty), H^2([0, 1])). \quad (1.24)$$

Guo etc. in [5] studied the Cauchy problem of (1.3) with initial data

$$(u^0, v^0)(x, 0) = (u_0^0(x), v_0^0(x)) \rightarrow (u_\pm, v_\infty) \text{ as } x \rightarrow \pm\infty. \quad (1.25)$$

They proved the existence of global solutions to the Cauchy problem of a hyperbolic-parabolic coupled system with large initial data, which is summarized in Lemma 2.4 later.

Notations: Throughout this paper, we denote positive constants by C may depends on T , but is independent of ε . Moreover, the character “ C ” may differ in different places. $L^p = L^p(\Omega)$

$(1 \leq p \leq \infty)$ denotes usual Lebesgue space with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty} = \sup_{x \in \Omega} |f(x)|.$$

$H^l(\Omega)$ ($l \geq 0$) denotes the usual l th-order Sobolev space with the norm

$$\|f\|_l = \left(\sum_{j=0}^l \|\partial_x^j f\|^2 \right)^{\frac{1}{2}},$$

where $\Omega = \mathbf{R}$ or $[0, 1]$, and $\|\cdot\| = \|\cdot\|_0 = \|\cdot\|_{L^2}$. For simplicity, $\|f(\cdot, t)\|_{L^p}$ and $\|f(\cdot, t)\|_l$ are denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$ respectively.

The rest of the paper is organized as follows. In Section 2, we study the Cauchy problem (1.1) and (1.2). For any fixed $\varepsilon > 0$, the global unique solvability are obtained. When diffusion parameter $\varepsilon \rightarrow 0^+$, the limit problem is considered and we show that the convergence rates in L^∞ -norm is of the order $O(\varepsilon)$. In Section 3, we study the initial-boundary value problem (1.1), (1.5) and (1.6). In the last Section 4, for the convenience of the readers, we use the appendix to give the derivation of system (1.1) and (1.3).

2 Cauchy problem

2.1 Global existence on the case of $\varepsilon > 0$

In this section, we are concerned with the global existence of large-amplitude H^2 solutions to the Cauchy problem (1.1), (1.2), which will be proven by continuing a unique local solution using a uniform-in- ε *a priori* estimate in the setting of large data. The construction on the local existence of the solutions is standard based on iteration argument and Fixed Point Theorem. Next, we will devoted ourself to obtaining some uniform-in- ε *a priori* estimates on the solution $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ to the Cauchy problem (1.1), (1.2) for all $\varepsilon > 0$. To this end, $v^\varepsilon(x, t)$ is supposed to satisfy the priori assumption

$$v^\varepsilon(x, t) > 0, \quad (x, t) \in \mathbf{R} \times [0, T] \quad (2.1)$$

for any fixed $T > 0$.

Lemma 2.1 (Basic energy estimate) *Assume that the assumptions listed in Theorem 1.1 are satisfied. Then there exists a positive constant C , independent of ε , such that*

$$\int_{\mathbf{R}} \eta dx + \int_0^t \int_{\mathbf{R}} \frac{(v_x^\varepsilon)^2}{v^\varepsilon} dx ds + \varepsilon \int_0^t \int_{\mathbf{R}} (u_x^\varepsilon)^2 dx ds = \int_{\mathbf{R}} \eta_0 dx. \quad (2.2)$$

Proof. As in [25, 5], it is easy to see that

$$\begin{cases} \eta(u^\varepsilon, v^\varepsilon) = \frac{1}{2}(u^\varepsilon)^2 + v^\varepsilon \ln \frac{v^\varepsilon}{v_\infty} - (v^\varepsilon - v_\infty), \\ q(u^\varepsilon, v^\varepsilon) = -u^\varepsilon v^\varepsilon \ln \frac{v^\varepsilon}{v_\infty} + \frac{2}{3}\varepsilon (u^\varepsilon)^3 \end{cases}$$

is an entropy-entropy flux pair to the hyperbolic system (1.1) normalized at $(0, v_\infty)$. Moreover, based on the definition of the entropy-entropy flux pair (see [20]), it is easy to verify that $(\eta(u^\varepsilon, v^\varepsilon), q(u^\varepsilon, v^\varepsilon))$ satisfies

$$\eta_t + q_x = \varepsilon u^\varepsilon u_{xx}^\varepsilon + v_{xx}^\varepsilon \ln \frac{v^\varepsilon}{v_\infty}. \quad (2.3)$$

Integrating the above identity with respect x and t over $\mathbf{R} \times [0, t]$, by using integration by parts we can get (2.2). This completes the proof of Lemma 2.1.

Lemma 2.2 (First order energy estimate) *Assume that the assumptions listed in Theorem 1.1 are satisfied. Then there exists a positive constant C , independent of ε , such that*

$$\begin{aligned} \|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} + \int_{\mathbf{R}} \left\{ (u_x^\varepsilon)^2 + (v_x^\varepsilon)^2 \right\} dx \\ + \int_0^t \int_{\mathbf{R}} \left\{ v^\varepsilon (u_x^\varepsilon)^2 + (v_{xx}^\varepsilon)^2 \right\} dx ds + \varepsilon \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \leq C. \end{aligned} \quad (2.4)$$

Here and hereafter, C denotes positive constant depends on T .

Proof. As in [5], notice that

$$u_{xt}^\varepsilon = v_{xx}^\varepsilon - \varepsilon \left[(u^\varepsilon)^2 \right]_{xx} + \varepsilon u_{xxx}^\varepsilon = v_t^\varepsilon - (u^\varepsilon v^\varepsilon)_x - \left[\varepsilon (u^\varepsilon)^2 \right]_{xx} + \varepsilon u_{xxx}^\varepsilon. \quad (2.5)$$

Multiplying (2.5) by $2u_x^\varepsilon$, then integrating the resulting equation with respect x and t over $\mathbf{R} \times [0, t]$, we have by exploiting some integrations by parts and Cauchy-Schwarz inequality that

$$\begin{aligned} & \int_{\mathbf{R}} (u_x^\varepsilon)^2 dx - \int_{\mathbf{R}} (u_{0x}^\varepsilon)^2 dx + 2 \int_0^t \int_{\mathbf{R}} v^\varepsilon (u_x^\varepsilon)^2 dx ds + 2\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \\ &= 2 \int_0^t \int_{\mathbf{R}} v_t^\varepsilon u_x^\varepsilon dx ds - 2 \int_0^t \int_{\mathbf{R}} u^\varepsilon v_x^\varepsilon u_x^\varepsilon dx ds - 2\varepsilon \int_0^t \int_{\mathbf{R}} (u_x^\varepsilon)^3 dx ds \\ &= \sum_{i=1}^3 I_i. \end{aligned} \quad (2.6)$$

I_1 - I_3 are estimated as follows:

$$\begin{aligned} I_1 &= -2 \int_{\mathbf{R}} (v_0^\varepsilon - v_\infty) u_{0x}^\varepsilon dx + 2 \int_{\mathbf{R}} (v^\varepsilon - v_\infty) u_x^\varepsilon dx - 2 \int_0^t \int_{\mathbf{R}} (v^\varepsilon - v_\infty) u_{xt}^\varepsilon dx ds \\ &= -2 \int_{\mathbf{R}} (v_0^\varepsilon - v_\infty) u_{0x}^\varepsilon dx + 2 \int_{\mathbf{R}} (v^\varepsilon - v_\infty) u_x^\varepsilon dx - 2 \int_0^t \int_{\mathbf{R}} (v^\varepsilon - v_\infty) v_{xx}^\varepsilon dx ds \\ &\quad + 2\varepsilon \int_0^t \int_{\mathbf{R}} (v^\varepsilon - v_\infty) \left[(u^\varepsilon)^2 \right]_{xx} dx ds - 2\varepsilon \int_0^t \int_{\mathbf{R}} (v^\varepsilon - v_\infty) u_{xxx}^\varepsilon dx ds \\ &= \sum_{i=1}^5 I_1^i. \end{aligned} \quad (2.7)$$

It is easy to get

$$I_1^1 \leq \int_{\mathbf{R}} (v_0^\varepsilon - v_\infty)^2 dx + \int_{\mathbf{R}} (u_{0x}^\varepsilon)^2 dx \leq C. \quad (2.8)$$

Notice that there exists a positive constant $C > 0$ such that

$$|v^\varepsilon - v_\infty| \leq C \left\{ v^\varepsilon \ln \frac{v^\varepsilon}{v_\infty} - (v^\varepsilon - v_\infty) \right\} \leq C \eta(u^\varepsilon, v^\varepsilon), \quad \text{for } v^\varepsilon \in \Omega_1 \quad (2.9)$$

and

$$|v^\varepsilon - v_\infty|^2 \leq C \left\{ v^\varepsilon \ln \frac{v^\varepsilon}{v_\infty} - (v^\varepsilon - v_\infty) \right\} \leq C \eta(u^\varepsilon, v^\varepsilon), \quad \text{for } v^\varepsilon \in \Omega_2, \quad (2.10)$$

since

$$\sup_{v^\varepsilon \in \Omega_1} \frac{|v^\varepsilon - v_\infty|}{v^\varepsilon \ln \frac{v^\varepsilon}{v_\infty} - (v^\varepsilon - v_\infty)} = \frac{1}{3 \ln \frac{3}{2} - 1}$$

and

$$\sup_{v^\varepsilon \in \Omega_2} \frac{|v^\varepsilon - v_\infty|^2}{v^\varepsilon \ln \frac{v^\varepsilon}{v_\infty} - (v^\varepsilon - v_\infty)} = \frac{v_\infty}{6 \ln \frac{3}{2} - 2}.$$

Here

$$\Omega_1 = \left\{ v^\varepsilon : |v^\varepsilon - v_\infty| \geq \frac{v_\infty}{2} \right\} \cap \{v^\varepsilon : v^\varepsilon > 0\} = \left\{ v^\varepsilon : 0 < v^\varepsilon \leq \frac{v_\infty}{2} \right\} \cup \left\{ v^\varepsilon \geq \frac{3v_\infty}{2} \right\}$$

and

$$\Omega_2 = \left\{ v^\varepsilon : |v^\varepsilon - v_\infty| \leq \frac{v_\infty}{2} \right\} = \left\{ v^\varepsilon : \frac{v_\infty}{2} \leq v^\varepsilon \leq \frac{3v_\infty}{2} \right\}.$$

Then we have from (2.9) and (2.10)

$$\begin{aligned} I_1^2 &\leq 2 \int_{\mathbf{R}} (v^\varepsilon - v_\infty)^2 dx + \frac{1}{2} \int_{\mathbf{R}} (u_x^\varepsilon)^2 dx \\ &= 2 \int_{\{x: v^\varepsilon \in \Omega_1\}} (v^\varepsilon - v_\infty)^2 dx + 2 \int_{\{x: v^\varepsilon \in \Omega_2\}} (v^\varepsilon - v_\infty)^2 dx + \frac{1}{2} \int_{\mathbf{R}} (u_x^\varepsilon)^2 dx \\ &\leq C \left(1 + \|v^\varepsilon - v_\infty\|_{L^\infty(\mathbf{R} \times [0, T])} \right) \int_{\mathbf{R}} \eta_0 dx + \frac{1}{2} \int_{\mathbf{R}} (u_x^\varepsilon)^2 dx. \end{aligned} \quad (2.11)$$

By using integration by part, Cauchy-Schwartz inequality and Lemma 2.1, we have

$$I_1^3 = 2 \int_0^t \int_{\mathbf{R}} (v_x^\varepsilon)^2 dx ds \leq \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \int_0^t \int_{\mathbf{R}} \frac{(v_x^\varepsilon)^2}{v^\varepsilon} dx ds, \quad (2.12)$$

$$\begin{aligned} I_1^4 &= -4\varepsilon \int_0^t \int_{\mathbf{R}} u^\varepsilon u_x^\varepsilon v_x^\varepsilon dx ds \\ &\leq \int_0^t \int_{\mathbf{R}} v^\varepsilon (u_x^\varepsilon)^2 dx ds + \|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} \frac{(v_x^\varepsilon)^2}{v^\varepsilon} dx ds \\ &\leq \int_0^t \int_{\mathbf{R}} v^\varepsilon (u_x^\varepsilon)^2 dx ds + C \|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} I_1^5 &= 2\varepsilon \int_0^t \int_{\mathbf{R}} v_x^\varepsilon u_{xx}^\varepsilon dx ds \\ &\leq \varepsilon \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds + \varepsilon \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \int_0^t \int_{\mathbf{R}} \frac{(v_x^\varepsilon)^2}{v^\varepsilon} dx ds. \end{aligned} \quad (2.14)$$

The estimate on I_2 is the same as those of I_1^4 .

Finally, we estimate I_3 . By using Gagliardo-Nirenberg inequality and Young inequality, it follows that

$$\|u_x^\varepsilon(t)\|_{L^3}^3 \leq C \|u_{xx}^\varepsilon(t)\|^{\frac{7}{4}} \|u^\varepsilon(t)\|^{\frac{5}{4}} \leq \frac{1}{4} \|u_{xx}^\varepsilon(t)\|^2 + C \|u^\varepsilon(t)\|^{10}. \quad (2.15)$$

Thus, we have from Lemma 2.1

$$\begin{aligned} I_3 &\leq \frac{1}{2}\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds + C\varepsilon \int_0^t \left\{ \int_{\mathbf{R}} (u^\varepsilon)^2 dx \right\}^5 ds \\ &\leq \frac{1}{2}\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds + C. \end{aligned} \quad (2.16)$$

Substituting (2.7)-(2.16) into (2.6), we get that

$$\begin{aligned} &\int_{\mathbf{R}} (u_x^\varepsilon)^2 dx + \int_0^t \int_{\mathbf{R}} v^\varepsilon (u_x^\varepsilon)^2 dx ds + \varepsilon \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \\ &\leq C(T) \left(1 + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} + \|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \right). \end{aligned} \quad (2.17)$$

From the Sobolev inequality and Cauchy inequality, we have

$$\|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \leq 2 \sup_{[0, T]} \{ \|u^\varepsilon(t)\| \|u_x^\varepsilon(t)\| \} \leq \sup_{[0, T]} \left\{ C \|u^\varepsilon(t)\|^2 + \frac{1}{2C} \|u_x^\varepsilon(t)\|^2 \right\}. \quad (2.18)$$

Substituting (2.18) into (2.17), using Lemma 2.1, we deduce immediately

$$\begin{aligned} &\int_{\mathbf{R}} (u_x^\varepsilon)^2 dx + \int_0^t \int_{\mathbf{R}} v^\varepsilon (u_x^\varepsilon)^2 dx ds + \varepsilon \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \\ &\leq C \left(1 + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \right). \end{aligned} \quad (2.19)$$

From Lemma 2.1, (2.18) and (2.19), we have

$$\|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \leq C \left(1 + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \right). \quad (2.20)$$

The rest of the proof of this lemma is similar to those in [5]. By differentiating the second equation of (1.1) with respect to x once and multiplying the resulting identity by $2v_x^\varepsilon$, then integrating the final equation with respect x and t over $\mathbf{R} \times [0, t]$, we have by integrations by parts, (2.18), (2.19) and Lemma 2.1 that

$$\int_{\mathbf{R}} (v_x^\varepsilon)^2 dx + \int_0^t \int_{\mathbf{R}} (v_{xx}^\varepsilon)^2 dx ds \leq C \left(1 + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \right). \quad (2.21)$$

In order to get the $L^\infty(\mathbf{R} \times [0, T])$ -norm estimate on $(u^\varepsilon(x, t), v^\varepsilon(x, t))$, take

$$h(v^\varepsilon) = v^\varepsilon \ln \frac{v^\varepsilon}{v_\infty} - (v^\varepsilon - v_\infty), \quad \varphi(v^\varepsilon) = \begin{cases} \int_{v_\infty}^{v^\varepsilon} \sqrt{h(z)} dz, & \text{for } v^\varepsilon \geq v_\infty, \\ \int_{v^\varepsilon}^{v_\infty} \sqrt{h(z)} dz, & \text{for } 0 < v^\varepsilon \leq v_\infty. \end{cases} \quad (2.22)$$

From (2.9) and (2.10), there exists a positive constant $C > 0$ such that

$$|v^\varepsilon - v_\infty| \leq C (h(v^\varepsilon) + 1),$$

which implies

$$|v^\varepsilon - v_\infty|^{\frac{1}{2}} \leq C \left(\sqrt{h(v^\varepsilon)} + 1 \right). \quad (2.23)$$

From (2.22) and (2.23), we have

$$\frac{2}{3} (v^\varepsilon - v_\infty)^{\frac{3}{2}} = \int_{v_\infty}^{v^\varepsilon} |z - v_\infty|^{\frac{1}{2}} dz \leq C \int_{v_\infty}^{v^\varepsilon} \sqrt{h(z)} dz + C (v^\varepsilon - v_\infty), \quad \text{for } v^\varepsilon \geq v_\infty \quad (2.24)$$

and

$$\frac{2}{3} (v_\infty - v^\varepsilon)^{\frac{3}{2}} = \int_{v^\varepsilon}^{v_\infty} |z - v_\infty|^{\frac{1}{2}} dz \leq C \int_{v^\varepsilon}^{v_\infty} \sqrt{h(z)} dz + C (v_\infty - v^\varepsilon), \quad \text{for } v^\varepsilon \leq v_\infty. \quad (2.25)$$

On the other hand, we have

$$\begin{aligned} \varphi(v^\varepsilon) &= \int_{-\infty}^x \varphi(v^\varepsilon)_y dy = \int_{-\infty}^x \sqrt{h(v^\varepsilon)} v_y^\varepsilon dy \\ &\leq \left\| \sqrt{h(v^\varepsilon)} \right\| \cdot \|v_x^\varepsilon\| \leq C \left(1 + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \right). \end{aligned} \quad (2.26)$$

Thus for any $v^\varepsilon > 0$ and some positive constant $C > 0$, we have

$$C \left(|v^\varepsilon - v_\infty|^{\frac{3}{2}} - |v^\varepsilon - v_\infty| \right) \leq \varphi(v^\varepsilon) \leq C \left(1 + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \right). \quad (2.27)$$

This means that

$$\|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \leq C, \quad (2.28)$$

and consequently we have from (2.20)

$$\|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \leq C. \quad (2.29)$$

Combination of (2.19), (2.21), (2.28) and (2.29) yields (2.4). This completes the proof of Lemma 2.2.

Remark 2.1. The assumption $(u_0^\varepsilon, v_0^\varepsilon - v_\infty) \in H^2(\mathbf{R})$ implies

$$\int_{\mathbf{R}} \eta_0 dx \leq C \left(\|u_0^\varepsilon\|^2 + \|v_0^\varepsilon - v_\infty\|^2 \right)$$

which and (2.4), (2.11) show that

$$\|v^\varepsilon(t) - v_\infty\|^2 \leq C \left(\|u_0^\varepsilon\|^2 + \|v_0^\varepsilon - v_\infty\|^2 \right).$$

At last, we deduce the $L^\infty([0, T], L^2(\mathbf{R}))$ -norm estimate on $(u_{xx}^\varepsilon(x, t), v_{xx}^\varepsilon(x, t))$.

Lemma 2.3 (Second-order energy estimate) *Assume that the assumptions listed in Theorem 1.1 are satisfied. Then there exists a positive constant C , independent of ε , such that*

$$\int_{\mathbf{R}} \left\{ (u_{xx}^\varepsilon)^2 + (v_{xx}^\varepsilon)^2 \right\} dx + \int_0^t \int_{\mathbf{R}} \left\{ \varepsilon (u_{xxx}^\varepsilon)^2 + (v_{xxx}^\varepsilon)^2 \right\} dx ds \leq C. \quad (2.30)$$

Proof. Differentiating (1.1) with respect to x , multiplying the resulting identity by $-2u_{xxx}^\varepsilon$ and $-2v_{xxx}^\varepsilon$ respectively, and integrating the adding result with respect x and t over $\mathbf{R} \times [0, t]$, we have from some integrations by parts that

$$\begin{aligned} & \int_{\mathbf{R}} \left\{ (u_{xx}^\varepsilon)^2 + (v_{xx}^\varepsilon)^2 \right\} dx + 2 \int_0^t \int_{\mathbf{R}} \left\{ \varepsilon (u_{xxx}^\varepsilon)^2 + (v_{xxx}^\varepsilon)^2 \right\} dx ds \\ &= \int_{\mathbf{R}} \left\{ (u_{0xx}^\varepsilon)^2 + (v_{0xx}^\varepsilon)^2 \right\} dx - 2 \int_0^t \int_{\mathbf{R}} v_{xxx}^\varepsilon (u^\varepsilon v^\varepsilon)_{xx} dx ds \\ & \quad + 2 \int_0^t \int_{\mathbf{R}} v_{xxx}^\varepsilon u_{xx}^\varepsilon dx ds - 2 \int_0^t \int_{\mathbf{R}} \left\{ \varepsilon (u^\varepsilon)^2 \right\}_{xxx} u_{xx}^\varepsilon dx ds \\ &= \sum_{i=4}^7 I_i. \end{aligned} \quad (2.31)$$

The estimates on I_5 - I_6 can be found in [5] as follow:

$$I_5 + I_6 \leq \int_0^t \int_{\mathbf{R}} (v_{xxx}^\varepsilon)^2 dx ds + C \left\{ 1 + \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \right\}. \quad (2.32)$$

Next we will be devoted to estimating I_7 .

$$\begin{aligned} I_7 &= 4\varepsilon \int_0^t \int_{\mathbf{R}} (u_x^\varepsilon)^2 u_{xxx}^\varepsilon dx ds + 4\varepsilon \int_0^t \int_{\mathbf{R}} u^\varepsilon u_{xx}^\varepsilon u_{xxx}^\varepsilon dx ds \\ &= \sum_{i=1}^2 I_7^i. \end{aligned} \quad (2.33)$$

From Cauchy inequality and Lemma 2.1, one has

$$\begin{aligned} I_7^1 &\leq \frac{1}{4}\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xxx}^\varepsilon)^2 dx ds + 16\varepsilon \int_0^t \int_{\mathbf{R}} (u_x^\varepsilon)^4 dx ds \\ &\leq \frac{1}{2}\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xxx}^\varepsilon)^2 dx ds + C \int_0^t \left\{ \int_{\mathbf{R}} (u^\varepsilon)^2 dx \right\}^7 ds \\ &\leq \frac{1}{2}\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xxx}^\varepsilon)^2 dx ds + C. \end{aligned} \quad (2.34)$$

Here we used the fact obtained by Gagliardo-Nirenberg inequality and Young inequality

$$\|u_x^\varepsilon(t)\|_{L^4}^4 \leq C \|u_{xxx}^\varepsilon(t)\|^{\frac{5}{3}} \|u^\varepsilon(t)\|^{\frac{7}{3}} \leq \frac{1}{64} \|u_{xxx}^\varepsilon(t)\|^2 + C \|u^\varepsilon(t)\|^{14}. \quad (2.35)$$

In addition, one has by Cauchy inequality and Lemma 2.2

$$\begin{aligned} I_7^2 &\leq \frac{1}{4}\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xxx}^\varepsilon)^2 dx ds + 16\varepsilon \|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \\ &\leq \frac{1}{4}\varepsilon \int_0^t \int_{\mathbf{R}} (u_{xxx}^\varepsilon)^2 dx ds + C \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds. \end{aligned} \quad (2.36)$$

Substituting (2.32)-(2.36) into (2.31), we have

$$\begin{aligned} & \int_{\mathbf{R}} \left\{ (u_{xx}^\varepsilon)^2 + (v_{xx}^\varepsilon)^2 \right\} dx + \int_0^t \int_{\mathbf{R}} \left\{ \varepsilon (u_{xxx}^\varepsilon)^2 + (v_{xxx}^\varepsilon)^2 \right\} dx ds \\ & \leq C \left\{ 1 + \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \right\}. \end{aligned} \quad (2.37)$$

(2.30) immediately follows by applying the Gronwall inequality. This completes the proof of Lemma 2.3.

Remark 2.2. By Sobolev inequality, Lemmas 2.2 and 2.3, we have

$$\|u_x^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \leq C \sup_{[0, T]} \left\{ \|u_x^\varepsilon(t)\|^{\frac{1}{2}} \|u_{xx}^\varepsilon(t)\|^{\frac{1}{2}} \right\} \leq C$$

and

$$\|v_x^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])} \leq C \sup_{[0, T]} \left\{ \|v_x^\varepsilon(t)\|^{\frac{1}{2}} \|v_{xx}^\varepsilon(t)\|^{\frac{1}{2}} \right\} \leq C,$$

which will be used later.

Combination of Lemmas 2.1-2.3 and Remark 2.1 yields to the uniform-in- ε *a priori* estimate (1.13), which implies the global existence of solutions by combining the local existence and uniqueness. Similar to [5], by standard maximal principle, we can prove (1.14) and the priori assumption (2.1) holds. This completes the proof of global existence in Theorem 1.1.

2.2 Convergence rate of zero viscosity limit

In this subsection, we turn to our another result, which is concerned with convergence rates of the vanishing diffusion viscosity. That is, we will give the proof of Theorem 1.2, and it suffices to show the following Lemma 2.6.

First, the global existence of the H^2 solutions to the Cauchy problem (1.3), (1.4) was included in [5], which is summarized as follows.

Lemma 2.4 (Global existence on the limit problem, cf. [5]) *Assume that the initial data $(u_0^0(x), v_0^0(x))$ satisfies*

$$\inf_{x \in \mathbf{R}} v_0^0(x) \geq \alpha > 0 \quad \text{and} \quad (u_0^0 - \bar{u}, v_0^0 - v_\infty) \in H^2(\mathbf{R}). \quad (2.38)$$

Then there exists a unique global solution $(u^0(x, t), v^0(x, t))$ of the Cauchy problem (1.3), (1.4) satisfying

$$(u^0 - \bar{u}, v^0 - v_\infty) \in L^\infty([0, \infty), H^2(\mathbf{R})), \quad v_x^0 \in L^2([0, \infty), H^2(\mathbf{R})). \quad (2.39)$$

Moreover, for any fixed $T > 0$, there exists a positive constant $C(T) > 0$ which depends only on T and $\|(u_0^0 - \bar{u}, v_0^0 - v_\infty)\|_{H^2}$ such that

$$v^0(x, t) \geq C(T). \quad (2.40)$$

Here the smooth monotone function $\bar{u}(x)$ satisfies $\bar{u}(x) = u_\pm$, $\pm x \geq 1$ for fixed constants u_- and u_+ .

Next, we need improve the regularity on the solutions to the zero viscosity limit problem (1.3), (1.4).

Lemma 2.5 (Regularity improved) *Assume that the assumptions listed in Theorem 1.2 are satisfied. Then there exists a positive constant C such that*

$$\int_{\mathbf{R}} \left\{ \left(u_{xxx}^0 \right)^2 + \left(v_{xxx}^0 \right)^2 \right\} dx + \int_0^t \int_{\mathbf{R}} \left(v_{xxxx}^0 \right)^2 dx ds \leq C. \quad (2.41)$$

Proof. Differentiating (1.3) with respect to x three times, multiplying the resulting identity by $2u_{xxx}^0$ and $2v_{xxx}^0$ respectively, and integrating the adding result with respect x and t over $\mathbf{R} \times [0, t]$, we have from some integrations by parts that

$$\begin{aligned} & \int_{\mathbf{R}} \left\{ \left(u_{xxx}^0 \right)^2 + \left(v_{xxx}^0 \right)^2 \right\} dx + 2 \int_0^t \int_{\mathbf{R}} \left(v_{xxxx}^0 \right)^2 dx ds \\ &= \int_{\mathbf{R}} \left\{ \left(u_{0xxx}^0 \right)^2 + \left(v_{0xxx}^0 \right)^2 \right\} dx - 2 \int_0^t \int_{\mathbf{R}} u^0 v_{xxx}^0 v_{xxxx}^0 dx ds - 6 \int_0^t \int_{\mathbf{R}} u_x^0 v_{xx}^0 v_{xxxx}^0 dx ds \\ & \quad - 6 \int_0^t \int_{\mathbf{R}} u_{xx}^0 v_x^0 v_{xxxx}^0 dx ds - 2 \int_0^t \int_{\mathbf{R}} u_{xxx}^0 v^0 v_{xxxx}^0 dx ds + 2 \int_0^t \int_{\mathbf{R}} u_{xxx}^0 v_{xxxx}^0 dx ds \\ &= \sum_{i=1}^6 J_i. \end{aligned} \quad (2.42)$$

J_2 - J_6 are estimated as follows:

$$J_2 \leq \frac{1}{5} \int_0^t \int_{\mathbf{R}} \left(v_{xxxx}^0 \right)^2 dx ds + C \left\| u^0 \right\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} \left(v_{xxx}^0 \right)^2 dx ds, \quad (2.43)$$

$$J_3 \leq \frac{1}{5} \int_0^t \int_{\mathbf{R}} \left(v_{xxxx}^0 \right)^2 dx ds + C \left\| u_x^0 \right\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} \left(v_{xx}^0 \right)^2 dx ds, \quad (2.44)$$

$$J_4 \leq \frac{1}{5} \int_0^t \int_{\mathbf{R}} \left(v_{xxxx}^0 \right)^2 dx ds + C \left\| v_x^0 \right\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} \left(u_{xx}^0 \right)^2 dx ds, \quad (2.45)$$

$$J_5 + J_6 \leq \frac{1}{5} \int_0^t \int_{\mathbf{R}} \left(v_{xxxx}^0 \right)^2 dx ds + C \left(1 + \left\| v^0 \right\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \right) \int_0^t \int_{\mathbf{R}} \left(u_{xxx}^0 \right)^2 dx ds. \quad (2.46)$$

Substituting (2.43)-(2.46) into (2.42), we get from Lemma 2.4

$$\begin{aligned} & \int_{\mathbf{R}} \left\{ \left(u_{xxx}^0 \right)^2 + \left(v_{xxx}^0 \right)^2 \right\} dx + \int_0^t \int_{\mathbf{R}} \left(v_{xxxx}^0 \right)^2 dx ds \\ & \leq C \left\{ 1 + \int_0^t \int_{\mathbf{R}} \left(u_{xxx}^0 \right)^2 dx ds \right\}. \end{aligned} \quad (2.47)$$

(2.41) immediately follows by applying the Gronwall inequality. This completes the proof of Lemma 2.5.

Based on Theorem 1.1 and Lemmas 2.4-2.5, the following L^2 -convergence rates can be proved.

Lemma 2.6 (L^2 -Convergence rates) *Assume that the assumptions listed in Theorem 1.2 are satisfied. Then there exists a positive constant C , independent of ε , such that*

$$\begin{aligned} & \int_{\mathbf{R}} \left[(u^\varepsilon - u^0)^2 + (v^\varepsilon - v^0)^2 \right] dx \\ & + \int_0^t \int_{\mathbf{R}} \left[\varepsilon (u^\varepsilon - u^0)_x^2 + (v^\varepsilon - v^0)_x^2 \right] dx d\tau \leq C\varepsilon^2 \end{aligned} \quad (2.48)$$

and

$$\begin{aligned} & \int_{\mathbf{R}} \left[(u^\varepsilon - u^0)_x^2 + (v^\varepsilon - v^0)_x^2 \right] dx \\ & + \int_0^t \int_{\mathbf{R}} \left[\varepsilon (u^\varepsilon - u^0)_{xx}^2 + (v^\varepsilon - v^0)_{xx}^2 \right] dx d\tau \leq C\varepsilon^2. \end{aligned} \quad (2.49)$$

Proof. Set

$$\psi^\varepsilon = u^\varepsilon - u^0, \quad \theta^\varepsilon = v^\varepsilon - v^0. \quad (2.50)$$

Then we deduce from (1.1)-(1.2) and (1.3)-(1.4) that $(\psi^\varepsilon, \theta^\varepsilon)(x, t)$ satisfy the following Cauchy problem:

$$\begin{cases} \psi_t^\varepsilon + \left(\varepsilon (u^\varepsilon)^2 - \theta^\varepsilon \right)_x = \varepsilon \psi_{xx}^\varepsilon + \varepsilon u_{xx}^0, \\ \theta_t^\varepsilon - (\psi^\varepsilon v^\varepsilon + u^0 \theta^\varepsilon)_x = \theta_{xx}^\varepsilon, \end{cases} \quad (2.51)$$

with initial data

$$(\psi^\varepsilon, \theta^\varepsilon)(x, 0) = (\psi_0^\varepsilon, \theta_0^\varepsilon). \quad (2.52)$$

Proof of (2.48).

Multiplying the first and second equation of (2.51) by $2\psi^\varepsilon$ and $2\theta^\varepsilon$ respectively, integrating the adding result with respect x and t over $\mathbf{R} \times [0, t]$, we have

$$\begin{aligned} & \int_{\mathbf{R}} \left\{ (\psi^\varepsilon)^2 + (\theta^\varepsilon)^2 \right\} dx + 2 \int_0^t \int_{\mathbf{R}} \left\{ \varepsilon (\psi_x^\varepsilon)^2 + (\theta_x^\varepsilon)^2 \right\} dx ds \\ & = \int_{\mathbf{R}} \left\{ (\psi_0^\varepsilon)^2 + (\theta_0^\varepsilon)^2 \right\} dx - 4\varepsilon \int_0^t \int_{\mathbf{R}} \psi^\varepsilon u^\varepsilon u_x^\varepsilon dx ds + 2\varepsilon \int_0^t \int_{\mathbf{R}} \psi^\varepsilon u_{xx}^0 dx ds \\ & \quad + 2 \int_0^t \int_{\mathbf{R}} \psi^\varepsilon \theta_x^\varepsilon dx ds - 2 \int_0^t \int_{\mathbf{R}} \psi^\varepsilon v^\varepsilon \theta_x^\varepsilon dx ds - 2 \int_0^t \int_{\mathbf{R}} u^0 \theta^\varepsilon \theta_x^\varepsilon dx ds \\ & = \sum_{i=7}^{12} J_i. \end{aligned} \quad (2.53)$$

Here J_7 - J_{12} are estimated as follows.

First, one has

$$J_7 \leq C\varepsilon^2 \quad (2.54)$$

since $\|u_0^\varepsilon - u_0^0\|_{H^2(\mathbf{R})} + \|v_0^\varepsilon - v_0^0\|_{H^2(\mathbf{R})} = O(\varepsilon)$.

Next, from Cauchy-Schwarz inequality, Theorem 1.1 and Lemma 2.4 with $\bar{u}(x) \equiv 0$, we obtain

$$\begin{aligned} J_8 & \leq 4\varepsilon^2 \|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (u_x^\varepsilon)^2 dx ds + \int_0^t \int_{\mathbf{R}} (\psi^\varepsilon)^2 dx ds \\ & \leq C\varepsilon^2 + \int_0^t \int_{\mathbf{R}} (\psi^\varepsilon)^2 dx ds, \end{aligned} \quad (2.55)$$

$$\begin{aligned}
J_9 &\leq 4\varepsilon^2 \int_0^t \int_{\mathbf{R}} (u_{xx}^0)^2 dx ds + \int_0^t \int_{\mathbf{R}} (\psi^\varepsilon)^2 dx ds \\
&\leq C\varepsilon^2 + \int_0^t \int_{\mathbf{R}} (\psi^\varepsilon)^2 dx ds
\end{aligned} \tag{2.56}$$

and

$$\begin{aligned}
\sum_{i=10}^{12} J_i &\leq 2 \left(1 + \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 + \|u^0\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \right) \int_0^t \int_{\mathbf{R}} \{(\psi^\varepsilon)^2 + (\theta^\varepsilon)^2\} dx ds \\
&\quad + \frac{3}{2} \int_0^t \int_{\mathbf{R}} (\theta_x^\varepsilon)^2 dx ds \\
&\leq C \int_0^t \int_{\mathbf{R}} \{(\psi^\varepsilon)^2 + (\theta^\varepsilon)^2\} dx ds + \frac{3}{2} \int_0^t \int_{\mathbf{R}} (\theta_x^\varepsilon)^2 dx ds.
\end{aligned} \tag{2.57}$$

Substituting (2.54)-(2.57) into (2.53), we get

$$\begin{aligned}
&\int_{\mathbf{R}} \{(\psi^\varepsilon)^2 + (\theta^\varepsilon)^2\} dx + \int_0^t \int_{\mathbf{R}} \{\varepsilon (\psi_{xx}^\varepsilon)^2 + (\theta_x^\varepsilon)^2\} dx ds \\
&\leq C\varepsilon^2 + C \int_0^t \int_{\mathbf{R}} \{(\psi^\varepsilon)^2 + (\theta^\varepsilon)^2\} dx ds.
\end{aligned} \tag{2.58}$$

We immediately get (2.48) from Gronwall inequality.

Proof of (2.49).

Multiplying the first and second equation of (2.51) by $-2\psi_{xx}^\varepsilon$ and $-2\theta_{xx}^\varepsilon$ respectively, integrating the adding result with respect x and t over $\mathbf{R} \times [0, t]$, we have

$$\begin{aligned}
&\int_{\mathbf{R}} \{(\psi_x^\varepsilon)^2 + (\theta_x^\varepsilon)^2\} dx + 2 \int_0^t \int_{\mathbf{R}} \{\varepsilon (\psi_{xx}^\varepsilon)^2 + (\theta_{xx}^\varepsilon)^2\} dx ds \\
&= \int_{\mathbf{R}} \{(\psi_{0x}^\varepsilon)^2 + (\theta_{0x}^\varepsilon)^2\} dx + 4\varepsilon \int_0^t \int_{\mathbf{R}} \psi_{xx}^\varepsilon u_x^\varepsilon \theta_x^\varepsilon dx ds - 2\varepsilon \int_0^t \int_{\mathbf{R}} \psi_{xx}^\varepsilon u_{xx}^0 dx ds \\
&\quad - 2 \int_0^t \int_{\mathbf{R}} \psi_{xx}^\varepsilon \theta_x^\varepsilon dx ds - 2 \int_0^t \int_{\mathbf{R}} (\psi^\varepsilon v^\varepsilon + u^0 \theta^\varepsilon)_x \theta_{xx}^\varepsilon dx ds \\
&= \sum_{i=13}^{17} J_i.
\end{aligned} \tag{2.59}$$

Here J_{13} - J_{17} are estimated as follows.

First, similar to the estimate (2.54), one has

$$J_{13} \leq C\varepsilon^2. \tag{2.60}$$

Next, from Cauchy-Schwarz inequality, Theorem 1.1, Lemmas 2.4-2.5, Remark 2.2 and (2.48), we obtain

$$\begin{aligned}
J_{14} &= -4\varepsilon \int_0^t \int_{\mathbf{R}} \psi_x^\varepsilon (u_x^\varepsilon)^2 dx ds - 4\varepsilon \int_0^t \int_{\mathbf{R}} \psi_x^\varepsilon u^\varepsilon v_{xx}^\varepsilon dx ds \\
&\leq C\varepsilon^2 \|u_x^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (u_x^\varepsilon)^2 dx ds + C\varepsilon^2 \|u^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (u_{xx}^\varepsilon)^2 dx ds \\
&\quad + \int_0^t \int_{\mathbf{R}} (\psi_x^\varepsilon)^2 dx ds \\
&\leq C\varepsilon^2 + \int_0^t \int_{\mathbf{R}} (\psi_x^\varepsilon)^2 dx ds,
\end{aligned} \tag{2.61}$$

$$\begin{aligned}
J_{15} &\leq \varepsilon^2 \int_0^t \int_{\mathbf{R}} (u_{xxx}^0)^2 dx ds + \int_0^t \int_{\mathbf{R}} (\psi_x^\varepsilon)^2 dx ds \\
&\leq C\varepsilon^2 + \int_0^t \int_{\mathbf{R}} (\psi_x^\varepsilon)^2 dx ds,
\end{aligned} \tag{2.62}$$

$$\begin{aligned}
J_{16} &= 2 \int_0^t \int_{\mathbf{R}} \psi_x^\varepsilon \theta_{xx}^\varepsilon dx ds \\
&\leq \int_0^t \int_{\mathbf{R}} (\psi_x^\varepsilon)^2 dx ds + \frac{1}{4} \int_0^t \int_{\mathbf{R}} (\theta_{xx}^\varepsilon)^2 dx ds
\end{aligned} \tag{2.63}$$

and

$$\begin{aligned}
J_{17} &= -2 \int_0^t \int_{\mathbf{R}} \psi_x^\varepsilon v^\varepsilon \theta_{xx}^\varepsilon dx ds - 2 \int_0^t \int_{\mathbf{R}} \psi^\varepsilon v_x^\varepsilon \theta_{xx}^\varepsilon dx ds - 2 \int_0^t \int_{\mathbf{R}} u_x^0 \theta^\varepsilon \theta_{xx}^\varepsilon dx ds \\
&\quad - 2 \int_0^t \int_{\mathbf{R}} u^0 \theta_x^\varepsilon \theta_{xx}^\varepsilon dx ds \\
&= \sum_{i=1}^4 J_{17}^i.
\end{aligned} \tag{2.64}$$

Here

$$\begin{aligned}
J_{17}^1 &\leq \frac{1}{4} \int_0^t \int_{\mathbf{R}} (\theta_{xx}^\varepsilon)^2 dx ds + C \|v^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (\psi_x^\varepsilon)^2 dx ds \\
&\leq \frac{1}{4} \int_0^t \int_{\mathbf{R}} (\theta_{xx}^\varepsilon)^2 dx ds + C \int_0^t \int_{\mathbf{R}} (\psi_x^\varepsilon)^2 dx ds,
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
J_{17}^2 &\leq \frac{1}{4} \int_0^t \int_{\mathbf{R}} (\theta_{xx}^\varepsilon)^2 dx ds + C \|v_x^\varepsilon\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (\psi^\varepsilon)^2 dx ds \\
&\leq \frac{1}{4} \int_0^t \int_{\mathbf{R}} (\theta_{xx}^\varepsilon)^2 dx ds + C\varepsilon^2,
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
J_{17}^3 + J_{17}^4 &\leq \frac{1}{4} \int_0^t \int_{\mathbf{R}} (\theta_{xx}^\varepsilon)^2 dx ds + C \|u_x^0\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (\theta^\varepsilon)^2 dx ds \\
&\quad + C \|u^0\|_{L^\infty(\mathbf{R} \times [0, T])}^2 \int_0^t \int_{\mathbf{R}} (\theta_x^\varepsilon)^2 dx ds \\
&\leq \frac{1}{4} \int_0^t \int_{\mathbf{R}} (\theta_{xx}^\varepsilon)^2 dx ds + C\varepsilon^2.
\end{aligned} \tag{2.67}$$

Substituting (2.60)-(2.67) into (2.59), we get

$$\begin{aligned}
&\int_{\mathbf{R}} \left\{ (\psi_x^\varepsilon)^2 + (\theta_x^\varepsilon)^2 \right\} dx + \int_0^t \int_{\mathbf{R}} \left\{ \varepsilon (\psi_{xx}^\varepsilon)^2 + (\theta_{xx}^\varepsilon)^2 \right\} dx ds \\
&\leq C\varepsilon^2 + C \int_0^t \int_{\mathbf{R}} \left\{ (\psi_x^\varepsilon)^2 + (\theta_x^\varepsilon)^2 \right\} dx ds.
\end{aligned} \tag{2.68}$$

We immediately get (2.49) from Gronwall inequality. This completes the proof of Lemma 2.6.

Finally, based on Lemma 2.6, we can prove Theorem 1.2. In fact, using Sobolev inequality, we also have from (2.48) and (2.49)

$$\begin{aligned} \left\| (u^\varepsilon - u^0)(t) \right\|_{L^\infty(\mathbf{R})} &\leq C \left\| (u^\varepsilon - u^0)(t) \right\|^{\frac{1}{2}} \left\| (u^\varepsilon - u^0)_x(t) \right\|^{\frac{1}{2}} \\ &\leq C\varepsilon \end{aligned} \quad (2.69)$$

and

$$\begin{aligned} \left\| (v^\varepsilon - v^0)(t) \right\|_{L^\infty(\mathbf{R})} &\leq C \left\| (v^\varepsilon - v^0)(t) \right\|^{\frac{1}{2}} \left\| (v^\varepsilon - v^0)_x(t) \right\|^{\frac{1}{2}} \\ &\leq C\varepsilon. \end{aligned} \quad (2.70)$$

This completes the proof of Theorem 1.2.

3 Initial-boundary value problem

In this section, we are concerned with the global existence of large-amplitude H^2 solutions to the initial-boundary value problem (1.1), (1.5) and (1.6). In particular, we investigate the convergence rates as the diffusion parameter ε goes to zero.

By the method similar to those in Subsection 2.1, we can prove (1.16) and (1.17) in Theorem 1.3 (i). Now we devoted ourselves to claiming (1.18) in Theorem 1.3 (i).

Proof of (1.18). From Sobolev inequality and (1.17), one has

$$\|u_x^\varepsilon\|_{L^\infty([0,1] \times [0,T])} \leq C.$$

Now set

$$M = \sup_{[0,1] \times [0,T]} \left\{ |u_x^\varepsilon(x, t)| \right\}, \quad w^\varepsilon(x, t) = e^{-\lambda t} v^\varepsilon(x, t),$$

where $\lambda > M$ is a positive constant.

We can deduce from (1.1), (1.5) and (1.6) that $w^\varepsilon(x, t)$ satisfies the following initial-boundary value problem

$$\begin{cases} w_t^\varepsilon - u^\varepsilon w_x^\varepsilon + (\lambda - u_x^\varepsilon) w^\varepsilon = w_{xx}^\varepsilon, & x \in (0, 1), \quad 0 < t < T, \\ w^\varepsilon(x, 0) = v_0^\varepsilon(x) \geq \alpha > 0, & x \in [0, 1], \\ w_x^\varepsilon(0, t) = w_x^\varepsilon(1, t) = 0, & 0 \leq t \leq T. \end{cases} \quad (3.1)$$

The standard maximal principle tells us that

$$w^\varepsilon(x, t) \geq 0, \quad (x, t) \in [0, 1] \times [0, T]. \quad (3.2)$$

In fact, if $w^\varepsilon(x, t)$ attains its negative minimum at some point $(x_0, t_0) \in [0, 1] \times (0, T]$. Thus $w^\varepsilon(x, t)$ satisfies

$$w_t^\varepsilon(x_0, t_0) \leq 0, \quad w_x^\varepsilon(x_0, t_0) = 0, \quad w^\varepsilon(x_0, t_0) < 0, \quad w_{xx}^\varepsilon(x_0, t_0) \geq 0,$$

which contradict to the first equation of (3.1) since $\lambda - u_x^\varepsilon > 0$.

One can deduce that $w^\varepsilon(x, t)$ satisfies from (3.1) and (3.2)

$$w_t^\varepsilon(x, t) + (\lambda + M)w^\varepsilon(x, t) - u^\varepsilon(x, t)w_x^\varepsilon(x, t) \geq w_{xx}^\varepsilon(x, t). \quad (3.3)$$

Let $H^\varepsilon(x, t) = e^{-\alpha t} (e^{(\lambda+M)t} w^\varepsilon(x, t) - \alpha)$, we have from (3.3) that

$$\begin{cases} H_t^\varepsilon(x, t) + (\lambda + M)H^\varepsilon(x, t) - u^\varepsilon(x, t)H_x^\varepsilon(x, t) \geq H_{xx}^\varepsilon(x, t), & x \in (0, 1), \quad 0 < t < T, \\ H^\varepsilon(x, 0) = w^\varepsilon(x, 0) - \alpha \geq 0, & x \in [0, 1], \\ H_x^\varepsilon(0, t) = H_x^\varepsilon(1, t) = 0, & 0 \leq t \leq T. \end{cases}$$

By the standard maximal principle, we have

$$H^\varepsilon(x, t) \geq 0, \quad (x, t) \in [0, 1] \times [0, T].$$

Thus for any $(x, t) \in [0, 1] \times [0, T]$, $v^\varepsilon(x, t)$ satisfies

$$v^\varepsilon(x, t) \geq \alpha e^{-Mt} \geq \alpha e^{-MT}. \quad (3.4)$$

This completes the proof of (1.18).

Similar results to those of Theorem 1.3 (i) can be obtained on the initial-boundary value problem on (1.8)-(1.11) as follows.

Lemma 3.1 *Let $(u_0^0, v_0^0) \in H^2([0, 1])$ and assume that there exists a positive constant $\alpha > 0$ such that $\inf_{x \in [0, 1]} v_0^0(x) \geq \alpha$. Then there exists a unique global solution $(u^0(x, t), v^0(x, t))$ of the initial-boundary value problem (1.8)-(1.11) satisfying*

$$\begin{cases} (u^0, v^0) \in L^\infty([0, \infty), H^2([0, 1])), \\ \frac{v_x^0}{\sqrt{v^0}} \in L^2([0, \infty), L^2([0, 1])), \quad v_{xx}^0 \in L^2([0, \infty), H^1([0, 1])). \end{cases} \quad (3.5)$$

Moreover, for any fixed $T > 0$, there exists a positive constant $C(T) > 0$ which depends only on T and $\|(u_0^0, v_0^0)\|_{H^2}$ such that

$$v^0(x, t) \geq C(T). \quad (3.6)$$

Based on Theorem 1.3 (i) and Lemma 3.1, the following L^2 -convergence rates can be proved.

Lemma 3.2 (L^2 -Convergence rates) *Assume that the assumptions listed in Theorem 1.3 are satisfied. Then there exists a positive constant C , independent of ε , such that*

$$\begin{aligned} & \int_0^1 \left[(u^\varepsilon - u^0)^2 + (v^\varepsilon - v^0)^2 \right] dx \\ & + \int_0^t \int_0^1 \left[\varepsilon (u^\varepsilon - u^0)_x^2 + (v^\varepsilon - v^0)_x^2 \right] dx d\tau \leq C\varepsilon^2 \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} & \int_0^1 \left[(u^\varepsilon - u^0)_x^2 + (v^\varepsilon - v^0)_x^2 \right] dx \\ & + \int_0^t \int_0^1 \left[\varepsilon (u^\varepsilon - u^0)_{xx}^2 + (v^\varepsilon - v^0)_{xx}^2 \right] dx d\tau \leq C\varepsilon. \end{aligned} \quad (3.8)$$

Proof. Set

$$\psi^\varepsilon = u^\varepsilon - u^0, \quad \theta^\varepsilon = v^\varepsilon - v^0. \quad (3.9)$$

Then we deduce from (1.1), (1.5), (1.6) and (1.8)-(1.11) that $(\psi^\varepsilon, \theta^\varepsilon)(x, t)$ satisfy the following initial-boundary value problem:

$$\begin{cases} \psi_t^\varepsilon + \left(\varepsilon (u^\varepsilon)^2 - \theta^\varepsilon \right)_x = \varepsilon \psi_{xx}^\varepsilon + \varepsilon u_{xx}^0, \\ \theta_t^\varepsilon - (\psi^\varepsilon v^\varepsilon + u^0 \theta^\varepsilon)_x = \theta_{xx}^\varepsilon, \end{cases} \quad (3.10)$$

with initial data

$$(\psi^\varepsilon, \theta^\varepsilon)(x, 0) = (\psi_0^\varepsilon, \theta_0^\varepsilon), \quad (3.11)$$

and the boundary conditions

$$(\psi^\varepsilon, \theta_x^\varepsilon)(0, t) = (\psi^\varepsilon, \theta_x^\varepsilon)(1, t) = (0, 0), \quad t \geq 0. \quad (3.12)$$

By the similar method to those in Subsection 2.2, we can obtain (3.7) and (3.8). We note that the convergence rate in (3.7) is the same as that shown in (2.48). However, the rate in (3.8) is slower than one obtained in (2.49), which is caused by the boundary effect. In fact, unlike Lemma 2.5, the regularity on solutions to the initial-boundary value problem (1.8)-(1.11) can not be improved. Consequently, the term $J_{15} = -2\varepsilon \int_0^t \int_0^1 \psi_{xx}^\varepsilon u_{xx}^0 dx ds$ (see (2.59) above) can only be controlled as follows:

$$\begin{aligned} -2\varepsilon \int_0^t \int_0^1 \psi_{xx}^\varepsilon u_{xx}^0 dx ds &\leq C\varepsilon \int_0^t \int_0^1 (u_{xx}^0)^2 dx ds + \varepsilon \int_0^t \int_0^1 (\psi_{xx}^\varepsilon)^2 dx ds \\ &\leq C\varepsilon + \varepsilon \int_0^t \int_0^1 (\psi_{xx}^\varepsilon)^2 dx ds. \end{aligned} \quad (3.13)$$

Finally, based on Lemma 3.2, we can prove Theorem 1.3 (ii). In fact, using Sobolev inequality, we also have from (3.7) and (3.8)

$$\begin{aligned} \left\| (u^\varepsilon - u^0)(t) \right\|_{L^\infty[0,1]} &\leq C \left\| (u^\varepsilon - u^0)(t) \right\|^{\frac{1}{2}} \left\| (u^\varepsilon - u^0)_x(t) \right\|^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{3}{4}} \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \left\| (v^\varepsilon - v^0)(t) \right\|_{L^\infty[0,1]} &\leq C \left\| (v^\varepsilon - v^0)(t) \right\|^{\frac{1}{2}} \left\| (v^\varepsilon - v^0)_x(t) \right\|^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{3}{4}}. \end{aligned} \quad (3.15)$$

This completes the proof of Theorem 1.3.

4 Appendix: Derivation of models

In this section, we give the derives of conservation laws (1.1) and (1.3) for the convenience of the readers respectively, cf. [14, 25, 5].

As in [14], one considers the case where $f(c) = \alpha c$ ($\alpha > 0$) in the Keller-Segel model (1.20), which means that oxygen is consumed only when cells (bacteria) encounter the chemical (oxygen). The crucial step in the derivation is to make a change of variable through the Hopf-Cole transformation

$$v = -c^{-1}c_x = -(\ln c)_x \quad (4.1)$$

which was first introduced in [21] for a chemotaxis model proposed in [9] describing the chemotactic movement for non-diffusible chemicals (i.e. $\varepsilon = 0$), and was later applied in [12, 13] to study the nonlinear stability of traveling wave solutions. It turns out this transformation also extends its capacity to the full Keller-Segel model (1.20) for $\varepsilon \neq 0$. Indeed, with the Hopf-Cole transformation (4.1), they derive the following viscous conservation laws from (1.20)

$$\begin{cases} u_t + (\varepsilon u^2 - \alpha v)_x = \varepsilon u_{xx}, \\ v_t - \chi(uv)_x = Dv_{xx}. \end{cases} \quad (4.2)$$

Substituting the scalings

$$\tilde{t} = \alpha t, \quad \tilde{x} = \sqrt{\frac{\alpha}{\chi}}x, \quad \tilde{v} = \sqrt{\frac{\alpha}{\chi}}v, \quad \tilde{D} = \frac{D}{\chi}, \quad \tilde{\varepsilon} = \frac{\varepsilon}{\chi} \quad (4.3)$$

into (4.2) and dropping the tildes for convenience, we obtain the system of conservation laws (1.1). By the transformation mentioned above, we get conservation laws (1.3) from (1.21).

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